①

(A) Differentiable in $[0,1]$.

Since, $\lim_{x \to 0^+} \frac{x^{3/2} \sin(1/x)}{x} = 0$

(B) f is absolutely continuous.

Since, f' is bounded & by mean value theorem,

$|f(x) - f(y)| \leq \sup_{a \in [0,1]} |f'(a)| \, |x-y|, \ \forall x, y \in [0,1]$

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②

(A) f is absolutely continuous

Since, $|x-y|^2 \leq |x-y|, \ \forall x, y \in [0,1]$

(B) False.

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③

(A) True

$|f(x) - f(y)| \leq |x-y| \Rightarrow m(f(A)) \leq m(A)$, for every & measurable set A. (or f is absolutely continuous)

$\therefore f$ maps measure zero sets, to measure zero set.

(B) False.

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④

(A) True.

$\int_I |f| \, dx \geq \left| \int_I f(x) \, dx \right| \geq I$.

$\Rightarrow \int_I (|f(x)| - 1) \, dx \geq 0$, for every interval $I \subseteq \mathbb{R}$.

$\Rightarrow \int_O (|f(x)| - 1) \, dx \geq 0$, for every open set $O \subseteq \mathbb{R}$.

Since, any measurable set A can be written as $A = O \sim E$, where O is an G_δ set & $m(E) = 0$.

$\Rightarrow \int_A (|f(x)|-1)\,dx \geq 0$, for every measurable set A

$\Rightarrow |f| \geq 1$ a.e

(B) Not true.

(5)

(A) True

~~It follows from~~

Let $E_N = \{x : |f(x)| \leq N\}$.

By MCT, $\exists$ $N>0$ $\ni$ $\int (|f| - f_N) < \varepsilon/2$ when $f_N = |f|\chi_{E_N}$

$\therefore$ Given $\varepsilon>0$ choose $\delta \ni N\delta < \varepsilon/2$.

$\therefore$ If $m(E)<\delta$, then $\int_E |f| = \int_E (|f| - f_N) + \int_E f_N$.

$$\leq \varepsilon/2 + N\,m(E)$$

$$< \varepsilon.$$

(B) True.

Since, $f \in L^1$, almost every $x \in \mathbb{R}$ is a Lebesgue point of f.

$\therefore g$ is differentiable a.e.

(6)

(A) False.

$(Mg)(x) \geq \frac{a}{2}\frac{1}{|x|}$, for a.e x. where $a = \sup_{0<y<1} |f(y)|$

(B) False.

By the same arguement as in (A).

(C) True.

$$(Mg)(x) = \begin{cases} \sup_{y\in(0,1)} \frac{1}{2(y-x)} \int_0^y |f(z)|\,dz & \text{if } x<0 \\ \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(z)|\,dz & \text{if } x \geq 0. \end{cases}$$

$\Rightarrow (Mg)(x) \leq a$, where $a = \sup_{0 \leq y \leq 1} |f(y)|$

$\Rightarrow Mg \in L^{\infty}(\mathbb{R})$.

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(7)

(A) True.

$f \in L^{p}(\mathbb{R}^n) \Rightarrow f \in L^{p}(Q)$

& Since $m(Q) = 1$, $f \in L^{q}(Q)$, $\forall$ q $1 \leq q \leq p$

$\Rightarrow g \in L^{q}(\mathbb{R}^n)$, for all $1 \leq q \leq p$.

~~By~~ By Theorem: 8.18 of "Real & Complex Analysis by Rudin"

$Mg \in L^{q}(\mathbb{R}^n)$, for all $1 \leq q \leq p$.

(B) Not true.

Let $n = 1$ & $p = 2$.

$f(x) = \chi_{[0,1]}(x) \dfrac{1}{x^{1/4}} \in L^{2}(\mathbb{R})$.

$g \in L^{1}(\mathbb{R}) \Rightarrow$ almoset every point is a Lebesgue point.

$\Rightarrow (Mg)(x) \geq |g(x)|$

$\Rightarrow Mg \notin L^{4}(\mathbb{R})$

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(8)

(A) True.

$$0 = (Mf)(0) = \sup_{r > 0} \frac{1}{m(B_r(0))} \int_{B_r(0)} |f(x)|\,dx \geq \frac{1}{m(B_s(0))} \int_{B_s(0)} |f(x)|\,dx , \ \forall s > 0.$$

$\Rightarrow \int_{B_s(0)} |f(x)|\,dx = 0$, $\forall s > 0$.

$\Rightarrow \int_{\mathbb{R}^n} |f|\,dm = 0 \Rightarrow f = 0$ a.e.

(B) not true

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(9)

(A) True.

$$\int_0^a f(x)\,dx = 0 \quad , \forall a \in (0,\infty) \Rightarrow \int_a^b f(x)\,dx = 0, \quad \text{of } 0 < a < b < \infty$$

$$\Rightarrow \int_A f\,dm = 0, \text{ for every measurable } A \subseteq (0,\infty).$$

$$\Rightarrow f = 0 \text{ a.e on } (0,\infty).$$

(B) Not true.

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(10)

(B) True

$$\text{If } \int_{-a}^{a} f(x)\,dx = 0 \quad , \forall a \in \mathbb{R}.$$

$$\Rightarrow \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = 0, \quad \forall a \in \mathbb{R}.$$

$$\Rightarrow \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = 0, \quad \forall a \in (0,\infty)$$

$$\Rightarrow \int_0^a [f(x) + f(-x)]\,dx = 0, \quad \forall a \in (0,\infty).$$

$\therefore$ By (9), $\cancel{f(x) = f(}$ $f(-x) = -f(x)$, $\forall$ ~~$x \in \mathbb{R}$~~. for a.e $x \in \mathbb{R}$.

f is an odd function a.e.

(A) Not true.

consider any odd function which is not zero a.e.

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